Hausdorff gaps reconstructed from Luzin gaps

by

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P. Simon - during his visit in Katowice (September 2008) - suggested that Hausdorff gaps and Luzin gaps do not look compatible. At first, we have tried to verify a such opinion.

M. Scheepers discerned something similar in [11]. Albeit, he wrote there that Luzin gaps are reminiscent of Hausdorff gaps, see comments at the end of the part "The Classical Era".

In [8], K. Kunen declared that "The easiest to construct are Luzin gaps" and that constructions of Hausdorff gaps need some stronger inductive hypotheses.

Usually, constructions of Hausdorff gaps and Luzin gaps are considered apart. Some Hausdorff gaps have been examined via topological manner, through gap spaces associated with them, for example [2], [3] or [9]. Forcing methods yield other treads to examine variety of Hausdorff gaps, for example [1], [6], [11] or [13].

Below, we try to compare Hausdorff and Luzin gaps.

Some definitions

- Sets A and B are *almost disjoint*, whenever $A \cap B$ is finite.
- A set A is *almost contained* in B, whenever $B \setminus A$ is finite.
- A family $\mathcal{Q}$ is called AD-*family*, whenever any two members of $\mathcal{Q}$ are almost disjoint.
- Families $\mathcal{Q}$ and $\mathcal{H}$ are called *separated*, if there exists a set C such that each member of $\mathcal{Q}$ is almost contained in C and each member of $\mathcal{H}$ is almost disjoint with C. [We may say that C separates $\mathcal{Q}$ from $\mathcal{H}$].

We use following conventions whenever A is almost contained in B. Namely, $A \subseteq^* B$ whenever $A \setminus B$ is finite, $A \subset^* B$ whenever $A \setminus B$ is finite and $B \setminus A$ is infinite.

Notions concerning gaps are restricted to families of infinite subsets of natural numbers, i.e. to subfamilies of $[\omega]^\omega$.

Definitions of gaps

- A pair of indexed families $[\{A_\alpha : \alpha < \omega_1\}; \{B_\alpha : \alpha < \omega_1\}]$ is called *Hausdorff pre-gap*, whenever $\alpha < \beta < \omega_1$ implies $A_\alpha \subset^* A_\beta \subset^* B_\beta \subset^* B_\alpha$.
- A Hausdorff pre-gap $[\{A_\alpha : \alpha < \omega_1\}; \{B_\alpha : \alpha < \omega_1\}]$ is called *Hausdorff gap*, whenever no set C almost contains any A_α and is almost contained in any B_α.
- An AD-family $\mathcal{Q}$ of the cardinality ω_1 is called *Luzin gap*, whenever no two disjoint uncountable subfamilies of $\mathcal{Q}$ are separated.

Following [11], a pair of families $[\mathcal{A}; \mathcal{B}]$ forms *Kunen pre-gap*, whenever $V \in \mathcal{A}$ and $U \in \mathcal{B}$ imply that the intersection $V \cap U$ is finite. A Kunen pre-gap $[\mathcal{A}; \mathcal{B}]$ is called *Kunen gap* whenever $\mathcal{A}$ and $\mathcal{B}$ are not separated.

Some observations

If one splits a Luzin gap onto two uncountable subfamilies, then one obtains a Kunen gap.

Suppose $[\{A_\alpha : \alpha < \omega_1\}; \{B_\alpha : \alpha < \omega_1\}] = \mathcal{H}$ is a Hausdorff pre-gap, then one can associate to it a few Kunen pre-gaps:

1. $[\{A_{\alpha+1} \setminus A_\alpha : \alpha < \omega_1\}; \{B_\alpha \setminus B_{\alpha+1} : \alpha < \omega_1\}]$;
2. $[\{A_\alpha : \alpha < \omega_1\}; \{B_\alpha \setminus B_{\alpha+1} : \alpha < \omega_1\}]$;
3. $[\{A_{\alpha+1} \setminus A_\alpha : \alpha < \omega_1\}; \{\omega \setminus B_\alpha : \alpha < \omega_1\}]$.
4. $[\{A_\alpha : \alpha < \omega_1\}; \{\omega \setminus B_\alpha : \alpha < \omega_1\}]$.

If one of above pre-gaps (1), (2), or (3) is a Kunen gap, then $\mathcal{H}$ has to be a Hausdorff gap. But, the pre-gap (4) is a Kunen gap iff $[\{A_\alpha : \alpha < \omega_1\}; \{B_\alpha : \alpha < \omega_1\}]$ is a Hausdorff gap. So, families (4) - but transformation of $\mathcal{H}$, we call *Hausdorff pre***−gap*.

Luzin gaps, a full discussion following [7]

Start with a pairwise disjoint family $\{A_n \in [\omega]^\omega : n \in \omega\}$. If almost disjoint sets $\{A_\beta : \beta < \alpha\}$ are just defined for $\alpha < \omega_1$, then enumerate them into a sequence $\{B_n : n \in \omega\}$. For every n, choose a set $\{d_1, d_2, \dots d_n\} \subset B_n \setminus (B_0 \cup B_1 \cup \dots \cup B_{n-1})$, with exactly n elements. Put A_α to be the union of all already chosen sets $\{d_1, d_2, \dots d_n\}$. The family $\{A_\alpha : \alpha < \omega_1\}$ is a Luzin gap.

Indeed, consider a partition of $\{A_\alpha : \alpha < \omega_1\}$ into two uncountably subfamilies $\mathcal{D}$ and $\mathcal{E}$. Suppose that a set set C separates $\mathcal{D}$ from $\mathcal{E}$. Fix a natural number n and uncountable subfamilies $\mathcal{F} \subseteq \mathcal{D}$ and $\mathcal{H} \subseteq \mathcal{E}$ such that $\cup\mathcal{F} \setminus n \subseteq C$ and $\cup\mathcal{H} \cap C \subseteq n$. Take $\alpha < \omega_1$ such that the intersection $\{A_\beta : \beta < \alpha\} \cap \mathcal{H}$ is infinite. Finally, for each $\gamma > \alpha$ with $A_\gamma \in \mathcal{F}$ there exist $\beta < \alpha$ and $A_\beta \in \mathcal{H}$ such that the intersection $A_\beta \cap A_\gamma$ is a set $\{d_1, d_2, \dots d_m\}$, where $m > n$. This is in conflict with $\cup\mathcal{F} \setminus n \subseteq C$ and $\cup\mathcal{H} \cap C \subseteq n$.

ZFC + $b > \omega_1$

In fact, the following lemma can be derived from Rothberger's Lemma 5 stated in [10].

A version of Rothberger's Lemma 5

Suppose a countable family $\mathcal{Q}$ consists of almost disjoint infinite subsets of natural numbers, and let $\mathcal{H}$ consists of sets almost disjoint with members of $\mathcal{Q}$. If $|\mathcal{H}| < b$, then families $\mathcal{Q}$ and $\mathcal{H}$ are separated. $\square$

The assumption $b > \omega_1$ is equivalent with Proposition (1): *The family of all sets of n.n. does not contain any* (Ω, ω^*) *gaps*; by Rothberger [10]. So, Rothberger carried out research on $b > \omega_1$.

Theorem (ZFC $+b > \omega_1$)

If $\{E_\alpha : \alpha < \omega_1\} \cup \{F_\alpha : \alpha < \omega_1\}$ is an AD-family, then there exists a Hausdorff pre-gap $[\{A_\alpha : \alpha < \omega_1\}; \{B_\alpha : \alpha < \omega_1\}]$ such that (always) $E_\alpha \subseteq^* A_{\alpha+1} \setminus A_\alpha \subseteq^* E_\alpha$ and $F_\alpha \subseteq^* B_\alpha \setminus B_{\alpha+1} \subseteq^* F_\alpha$.

Corollary (ZFC $+ b > \omega_1$)

If $\{E_\alpha : \alpha < \omega_1\} \cup \{F_\alpha : \alpha < \omega_1\}$ is a Luzin gap, then there exists a Hausdorff gap $[\{A_\alpha : \alpha < \omega_1\}; \{B_\alpha : \alpha < \omega_1\}]$ such that (always) $E_\alpha \subseteq^* A_{\alpha+1} \setminus A_\alpha \subseteq^* E_\alpha$ and $F_\alpha \subseteq^* B_\alpha \setminus B_{\alpha+1} \subseteq^* F_\alpha$. $\square$

If $b > \omega_1$ and there exists a Lebesgue non-measurable set of the cardinality ω_1, then there exist AD-families of the cardinality ω_1 which are non-measurable sets with respect to some Borel measures. Any family of sets which consists of a Hausdorff gap has to be universally measure zero.

HAUSDORFF PRE-GAPS AND ASSOCIETED TO THEM KUNEN PRE-GAPS HAVE DIFFERENT BOREL MEASURE PROPERTIES.

Proof of the theorem. We shall construct a desired Hausdorff pre-gap, defining by induction sets A_α and B_α such that

1. If $\beta < \alpha$, then $A_\beta \subset^* A_\alpha \subset^* B_\alpha \subset^* B_\beta$;
2. If $\alpha = \beta + 1$, then $E_\beta \cup A_\beta = A_\alpha$ and $B_\alpha = B_\beta \setminus F_\beta$;
3. Each member of the union $\{E_\beta : \alpha \leq \beta\} \cup \{F_\beta : \alpha \leq \beta\}$ is almost disjoint with A_α;
4. Each member of $\{E_\beta : \alpha \leq \beta\} \cup \{F_\beta : \alpha \leq \beta\}$ is almost contained in B_α.

Put $A_0 = \emptyset$ and $B_0 = \omega$ and $A_{\alpha+1} = E_\alpha \cup A_\alpha$ and $B_{\alpha+1} = B_\alpha \setminus F_\alpha$. It remains to define sets A_α and B_α for limit ordinals α. Take a sequence of ordinals $\gamma_0, \gamma_1, \dots$ which is increasing and has the limit α. Assume that $\gamma_0 = 0$.

At the first step, apply Rothberger's Lemma to families $\mathcal{Q} = \{A_{\gamma_{n+1}} \setminus A_{\gamma_n} : n \in \omega\}$ and $\mathcal{H} = \{B_{\gamma_n} \setminus B_{\gamma_{n+1}} : n \in \omega\} \cup \{E_\beta : \alpha \leq \beta\} \cup \{F_\beta : \alpha \leq \beta\}$. Take as A_α a set which separates $\mathcal{Q}$ from $\mathcal{H}$. Observe that $\beta < \alpha$ implies $A_\beta \subset^* A_\alpha \subset^* B_\beta$. Indeed, $\emptyset = A_{\gamma_0} \subset^* A_\alpha \subset^* B_{\gamma_0} = \omega$. Inductively, $A_{\gamma_n} \subseteq^* (A_{\gamma_n} \setminus A_{\gamma_{n-1}}) \cup A_{\gamma_{n-1}} \subset^* A_\alpha$, since A_α separates $\mathcal{Q}$ from $\mathcal{H}$. There exists $\gamma_n > \beta$, hence $A_\beta \subset^* A_{\gamma_n} \subset^* A_\alpha$. Also, one can assume that $A_\alpha \subset^* B_{\gamma_m}$. But sets A_α and $B_{\gamma_m} \setminus B_{\gamma_{m+1}}$ are almost disjoint, hence $A_\alpha \subset^* B_{\gamma_{m+1}}$. This gives that $A_\alpha \subset^* B_\beta$.

To define B_α apply Rothberger's Lemma to families $\mathcal{Q} = \{B_{\gamma_n} \setminus B_{\gamma_{n+1}} : n \in \omega\}$ and $\mathcal{H} = \{A_\alpha\} \cup \{E_\beta : \alpha \leq \beta\} \cup \{F_\beta : \alpha \leq \beta\}$. Take as B_α the complement of a set which separates $\mathcal{Q}$ from $\mathcal{H}$. The union $\{B_\alpha\} \cup \{B_{\gamma_n} \setminus B_{\gamma_{n+1}} : n \in \omega\}$ is an AD-family, hence $\beta < \alpha$ implies $B_\alpha \subset^* B_\beta$. Other induction conditions directly follow from the definitions. $\square$

ZFC + $(b = \omega_1)$

Let $\mathcal{Q} \subset [\omega]^\omega$ be an AD-family and $\mathcal{P} \subset [\omega]^\omega$.

Definition

$\mathcal{Q}$ is *almost disjoint refinement* of $\mathcal{P}$ (briefly $\mathcal{Q}$ is ADR of $\mathcal{P}$), whenever there exists a bijection $f : \mathcal{Q} \to \mathcal{P}$ such that $X \subseteq^* f(X)$ for every $X \in \mathcal{Q}$.

Now assume that $\mathcal{Q} = \mathcal{F} \cup \{B_n : n < \omega\}$ is an AD-family such that: $B_n = \{(n, k) : k < \omega\}$ for every $n < \omega$; and an unbounded family $\mathcal{F} = \{f_\alpha : \alpha < b\}$ consists of almost disjoint functions $f_\alpha : \omega \to \omega$ which are increasing. We have the following proposition under the hypothesis $b = \omega_1$.

Proposition (ZFC + $b = \omega_1$)

The family $\mathcal{Q}$ is ADR of no Hausdorff pre$^*-$gap.

Proof. Suppose that $\mathcal{Q}$ is ADR of a Hausdorff pre$^*-$gap $\{A_\alpha : \alpha < \omega_1\} \cup \{C_\alpha : \alpha < \omega_1\}$. Without loss of generality, we can fix α such that C_α almost contains infinitely many B_n. Thus the family

$$\mathcal{H} = \{f_\beta \in \mathcal{F} : f_\beta \subset^* \omega \times \omega \setminus C_\alpha\}$$

is uncountable and unbounded.

On the other hand, let $h(n) = \max\{k : (n,k) \notin C_\alpha\}$ whenever $B_n \subset^* C_\alpha$. Let $k_0, k_1, \ldots$ be an increasing enumeration of all n such that $B_n \subset^* C_\alpha$. Put $g(i) = h(k_n)$ whenever $k_{n-1} < i \leq k_n$. Because of $\mathcal{H}$ consists of increasing functions, one can check that g dominates any function from $\mathcal{H}$; a contradiction. □

NOW, WE SHOULD USE $b = \omega_1$ TO CONSTRUCT OF GAPS WITH SOME THINK UP PROPERTIES ...

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